Closed-Form Asymptotic Extraction Method for Coupled Microstrip Lines

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***Abstract*— The effective dielectric constants of a symmetric coupled microstrip line operating in the dominant mode are solved by using a closed-form formula of the asymptotic part of the impedance matrix elements. Using this asymptotic closed-form solution, the finite upper limit for numerical integration in the evaluation of the impedance matrix elements is significantly reduced. This results in improved computational efficiency for determining the $\epsilon_{\rm reff}$ of a coupled microstrip line while retaining a good accuracy compared to the conventional spectral domain approach (SDA).**

***Index Terms*—Asymptotic extraction, coupled microstrip lines.**

I. Introduction

THE COUPLED microstrip line has a variety of applications including couplers, filters in monolithic microwave integrated circuits (MMIC's), and crosstalk analysis in high-speed digital VLSI's. In analyzing a coupled microstrip line, the spectral domain approach (SDA) based on the rigorous full-wave method is primarily used. However, the full-wave method in the SDA is very time consuming because the evaluations of the impedance matrix elements include infinite integrals whose integrands have a slow convergent and highly oscillatory behavior [1].

Recently, a closed-form extraction technique of the asymptotic part of impedance matrix was introduced to solve the dispersion characteristics of single conductor open microstrip lines [1]. This method provides more accurate results and dramatically reduces the computation time than the conventional SDA. As an extension of the work in [1], the asymptotic part of impedance matrix in symmetric coupled microstrip line is also solved analytically by using the square root-weighted Chebyshev polynomial basis functions and the asymptotic behavior of the Green's function. Making use of this asymptotic closed-form solution, it is demonstrated that the proposed method significantly reduces the computation time, while retaining a good accuracy, for obtaining the effective dielectric constants for even and odd modes of symmetric coupled microstrip lines operating in the dominant mode.

II. Formulation

To solve for the phase constant β [and $\epsilon_{\rm reff}(f)$] in symmetric coupled microstrip lines, the even and odd modes method was used. In this method wave propagation along a coupled pair of strip conductor lines is expressed in terms of the even and odd modes by placing a magnetic wall and an electric wall, respectively, at the plane of symmetry. Because the center of strip conductor lines is now placed at $x = \pm(s+w)/2$ (where s is the spacing between two conductors and w is the width of each conductor) instead of at the origin, the currents in the spectral domain must be multiplied by $e^{\mp j(s+w)/2}$. The current density distributions on the strips are expanded in terms of a known set of square-weighted Chebyshev polynomial basis functions with unknown coefficients (a_m and b_n). By applying then Galerkin's method and Parseval's theorem [2], a system of coupled linear equations is obtained in matrix form as

$$\begin{bmatrix} [Z_{mm}] & [Z_{mn}] \\ [Z_{nm}] & [Z_{nn}] \end{bmatrix} \begin{bmatrix} [a_m] \\ [b_n] \end{bmatrix} = \begin{bmatrix} [0] \\ [0] \end{bmatrix} \quad \begin{matrix} m = 0, \cdots, M-1 \\ n = 0, \cdots, N-1 \end{matrix}. \tag{1}$$

The matrix elements in the coupled microstrip line are defined as

$$Z_{mm} = \int_{-\infty}^{\infty} \tilde{J}_{zm}(\alpha)\tilde{G}_{zz}(\alpha,\beta)\tilde{J}^*_{zm}(\alpha)S(\alpha)\,d\alpha \tag{2}$$

$$Z_{mn} = \int_{-\infty}^{\infty} \tilde{J}_{xn}(\alpha)\tilde{G}_{zx}(\alpha,\beta)\tilde{J}^*_{zm}(\alpha)S(\alpha)\,d\alpha \tag{3}$$

$$Z_{nm} = \int_{-\infty}^{\infty} \tilde{J}_{zm}(\alpha)\tilde{G}_{xz}(\alpha,\beta)\tilde{J}^*_{xn}(\alpha)S(\alpha)\,d\alpha \tag{4}$$

$$Z_{nn} = \int_{-\infty}^{\infty} \tilde{J}_{xn}(\alpha)\tilde{G}_{xx}(\alpha,\beta)\tilde{J}^*_{xn}(\alpha)S(\alpha)\,d\alpha \tag{5}$$

with

$$\tilde{J}_{zm}(\alpha) = (-j)^m \frac{\pi w}{2} J_m\left(\frac{w\alpha}{2}\right) \tag{6}$$

$$\tilde{J}_{xn}(\alpha) = (-j)^n \frac{\pi(n+1)}{\alpha} J_{n+1}\left(\frac{w\alpha}{2}\right) \tag{7}$$

$$S(\alpha) = [1 \pm e^{-j(s+w)\alpha}], \begin{cases} +, & \text{if even mode} \\ -, & \text{if odd mode} \end{cases} \tag{8}$$

where $J_n(\alpha)$ is the Bessel function of the first kind.

The generalized dyadic Green's function is used, which is obtained by recursive formulas [2]. To find the propagation constant β, which determines the dispersive characteristics of coupled microstrip line, Muller's root-finding method is utilized.

In order to accelerate the computation time for evaluating the matrix elements, the asymptotic extraction technique is

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employed as in [1]. In other words, the asymptotic part of the Green's function is subtracted from the original Green's function and added back, and the integral of (2), for example, is split into two parts:

$$Z_{mm} \simeq \int_{-\alpha_u}^{\alpha_u} \tilde{J}_{zm}(\alpha)\left[\tilde{G}_{zz}(\alpha,\beta) - \tilde{G}_{zz}^{\infty}(\alpha,\beta)\right]\tilde{J}_{zm}^{*}(\alpha) \times S(\alpha)d\alpha + \int_{-\infty}^{\infty} \tilde{J}_{zm}(\alpha)\tilde{G}_{zz}^{\infty}(\alpha,\beta)\tilde{J}_{zm}^{*}(\alpha)S(\alpha)\,d\alpha. \quad (9)$$

All other impedance matrix elements Z_{mn}, Z_{nm}, and Z_{nn} of (3)–(5) are treated in a similar manner.

In the second term of $S(\alpha)$ in (8), only the cosine term or the sine term can be survived due to the even and odd properties of the integrand. By using the asymptotic behaviors of the Green's functions, which were defined in [1, eqs. (4)–(6)], the asymptotic part of impedance matrix elements [the second term of (9)] of Z_{mm}, Z_{mn}, Z_{nm}, Z_{nn} has the general form of

$$Z^{\text{Asy}} = C_{mn}\int_{-\infty}^{\infty} \frac{J_m\left(\frac{w\alpha}{2}\right)J_n\left(\frac{w\alpha}{2}\right)}{\alpha}\{1 \pm e^{-j(x_i\alpha)}\}\,d\alpha, \quad \begin{cases} + & \text{if even mode} \\ - & \text{if odd mode} \end{cases} \quad (10)$$

where C_{mn} is constant and x_i represents $(s+w)$.

The numerical and analytical methods to evaluate the first integrals of (9) and (10) are carried out by using the same procedures mentioned in [1]. Therefore, in this letter, we concentrate on the evaluation of the second integral of (10).

III. CLOSED-FORM TECHNIQUE

Let us introduce a closed-form technique to evaluate the second integral of (10), I_{mn}, which is defined as

$$I_{mn} = \int_{0}^{\infty} \frac{J_m\left(\frac{w\alpha}{2}\right)J_n\left(\frac{w\alpha}{2}\right)}{\alpha} e^{-j(x_i\alpha)}\,d\alpha. \quad (11)$$

Using [3, eq. (5.43.1)], the products of the two Bessel functions can be expressed in integral form as

$$J_m(z)J_n(z) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} J_{m+n}(2z\cos\theta)\cos[(m-n)\theta]\,d\theta. \quad (12)$$

By inserting (12) into (11), it can be rewritten as

$$I_{mn} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\infty} \frac{J_{m+n}(w\,\alpha\,\cos\theta)}{\alpha} e^{-j(x_i\alpha)}\,d\alpha \times \cos[(m-n)\theta]\,d\theta. \quad (13)$$

To solve (13), first let us introduce [4, eqs. (6.693.1), (2)]:

$$\int_0^{\infty} \frac{J_\nu(a\alpha)e^{-j(x_i\alpha)}}{\alpha}\,d\alpha = \frac{a^\nu e^{-j(\frac{\nu\pi}{2})}}{\nu\left(x_i + \sqrt{x_i^2 - a^2}\right)^\nu} \quad (14)$$

where $(x_i \geq a)$ and $\text{Re}(\nu) > 0$ for the cosine term and $\text{Re}(\nu) > -1$ for the sine term.

After integrating (13) with respect to α by using (14), (13) can be rewritten as

$$I_{mn} = \frac{2}{\pi}\left(\frac{w}{x_i}\right)^{m+n} \frac{e^{-j(\frac{m+n}{2}\pi)}}{(m+n)} \times \int_0^{\frac{\pi}{2}} (\cos\theta)^{m+n}\cos[(m-n)\theta] \times \left[1 + \sqrt{1 - \left(\frac{w}{x_i}\cos\theta\right)^2}\,\right]^{-(m+n)} d\theta. \quad (15)$$

The last term of (15) can be expanded in series form by using the following Taylor series:

$$\frac{1}{(1+\sqrt{1-x^2})^q} = \frac{q}{2^q}\left\{\sum_{p=0}^{\infty}\binom{q+2p}{p}\left(\frac{1}{q+2p}\right)\left(\frac{x^2}{4}\right)^p\right\} \quad (16)$$

where q may be any real number and $x^2 < 1$.

By inserting (16) into (15), we obtain

$$I_{mn} = \frac{2}{\pi}\left(\frac{w}{x_i}\right)^{m+n}\frac{e^{-j(\frac{m+n}{2}\pi)}}{2^{m+n}} \times \left\{\sum_{p=0}^{\infty}\frac{\left(\frac{w}{2x_i}\right)^{2p}}{(m+n+2p)}\binom{m+n+2p}{p} \times \int_0^{\frac{\pi}{2}}(\cos\theta)^{m+n+2p}\cos[(m-n)\theta]\,d\theta\right\}. \quad (17)$$

The integral (17) with respect to θ can be solved explicitly by using [3, eq. (5.43)]

$$\int_0^{\frac{\pi}{2}}(\cos\theta)^{m+n+2p}\cos[(m-n)\theta]\,d\theta = \frac{\pi(m+n+2p)!}{2^{m+n+2p+1}(m+p)!(n+p)!}. \quad (18)$$

Substituting (18) into (17), I_{mn} of (11) can finally be written as

$$I_{mn} = \left(\frac{w}{4x_i}\right)^{m+n} e^{-j(\frac{m+n}{2}\pi)} \times \left\{\sum_{p=0}^{\infty}\frac{1}{(m+n+2p)}\binom{m+n+2p}{p}\left(\frac{w}{4x_i}\right)^{2p} \times \frac{(m+n+2p)!}{(m+p)!(n+p)!}\right\} \quad (19)$$

provided $w/x_i < 1$ and $(m+n) > 0$ for the cosine term, and $(m+n) > -1$ for the sine term.

The above series is very highly convergent if $w/x_i < 0.5$, and the first few terms (typically less than five) are sufficient to evaluate (19) accurately. If w/x_i ranges from 0.5 to 0.9, 5 to 30 series terms are used, depending on the values of w/x_i, m, and n. However, when $m = n = 0$ [as for matrix element Z_{00}, only the cosine term is survived in the integrand of (11) due to the even properties], the closed-form solution of (19) cannot be used due to the improper integral. In that case the integration of (11) is performed numerically using (11) of [1].

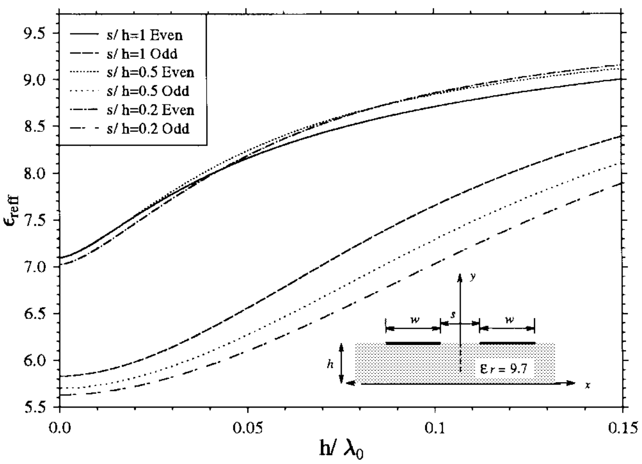

Fig. 1. Effective dielectric constants versus h/λ_0 for $s/h = 1$, $s/h = 0.5$, $s/h = 0.2$ ($w/h = 1$, $\epsilon_r = 9.7$, $\mu_r = 1$, $w = 1$ mm).

IV. Numerical Results and Conclusion

The effective dielectric constants obtained by using the closed-form solution of (19) and [1, eqs. (9), (11), and (18)] are plotted on Fig. 1 for $s/h = 0.2, 0.5, 1$ ($\epsilon_r = 9.7$, $w/h = 1$). We compared our results with those of Kowalski *et al.* [5] using the conventional SDA. They are in excellent agreement with each other. In this letter, the number of the basis functions $M = 9$ and $N = 8$ are used to describe the longitudinal and transverse current densities, respectively.

The effective dielectric constant in coupled microstrip lines is insensitive to the upper limit α_u for all the cases considered in this paper, if $\alpha_u > 30$ (rad/mm). Therefore, the upper limit $\alpha_u = 30$ (rad/mm) is used. To compare the computation time between the conventional SDA without the asymptotic extraction technique [upper limit $\alpha_u = 1000$ (rad/mm)] and the proposed method, the effective dielectric constants of the even mode and odd mode are listed in Table I for $\epsilon_r = 9.7$, $w/h = 1, s/h = 1$, and $h/\lambda_0 = 0.1$. All the results (ϵ_{reff}) of the four cases shown in Table I converge with an accuracy of 10^{-4} after the seventh iteration, starting from the quasi-TEM effective dielectric constant. As shown in Table I, the proposed method reduces the computation time by 20 times for both the even and odd modes compared to the conventional SDA.

In this paper, all the results (ϵ_{reff}) are shown for a thickness of $h = 1$ mm. But it should be noted that the asymptotic extraction technique does work very well down to a thickness of $h = 0.2$ mm with a upper limit $\alpha_u = 50$ (rad/mm). This value of upper limit reduces the computation time by a factor similar to that listed in Table I. As the substrate gets thinner ($h < 0.1$ mm), to obtain a desired accuracy it is necessary a greater upper limit α_u.

So far, we have demonstrated that the proposed method significantly reduces the CPU time over the conventional SDA to determine the even and odd mode ϵ_{reff}s of a symmetric coupled microstrip line operating in the dominant mode. The technique proposed in this letter can also be applied to coplanar waveguides.

TABLE I
Computer Time on a Silicon Graphic Indigo for the Calculation of the Effective Dielectric Constant with Two Different Techniques in the Coupled Microstrip Line ($h/\lambda_0 = 0.1$, $\epsilon_r = 9.7$, $\mu_r = 1$, $w = 1$ mm)

	SDA without asymptotic technique$^{(a)}$ ($\alpha_u = 1000(rad/mm)$)	Proposed Method$^{(b)}$ ($\alpha_u = 30(rad/mm)$)
Even mode (ϵ_{reff})	42.13 seconds (8.7381)	2.17 seconds (8.7173)
Odd mode (ϵ_{reff})	42.14 seconds (7.6247)	2.17 seconds (7.6391)
Computational Efficiency $\left(\frac{a}{b}\right)$: 19.42		

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References

[1] S. O. Park and C. A. Balanis, "Dispersion characteristics of open microstrip lines using closed-form asymptotic extraction," to be published in *IEEE Trans. Microwave Theory Tech.*, Mar. 1997.

[2] J. P. Gilb and C. A. Balanis, "Pulse distortion on multilayer coupled microstrip lines," *IEEE Trans. Microwave Theory Tech.*, vol. 37, pp. 1620–1628, Oct. 1989.

[3] G. N. Watson, *A Treatise on the Theory of Bessel Functions*. Cambridge, U.K.: Cambridge University Press, 1962.

[4] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*. New York: Academic, 1980.

[5] G. Kowalski and R. Pregla, "Dispersion characteristics of single and coupled microstrips," *Arch. Elek. {Ü}bertrang.*, vol. 26, pp. 276–280, 1972.